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P.I. RINGS AND THE LOCALIZATION AT HEIGHT 1 PRIME IDEALS

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ABSTRACT

Let R be a prime P.I. ring, finitely generated over a central noetherian subring. Let P be a height one prime ideal in R. We establish a finite criteria for the left (right) Ore localizability of P, provided P/P^2 is left (right) finitely generated. This replaces the noetherian assumption on R appearing in [BW], using an entirely different technique.

Introduction and notations

The present paper should be considered as a continuation of [BW]. Here we extend one of the main results of [BW] to the non-noetherian case. The present technique, however, is entirely different and the paper is independent of [BW].

In order to describe our main result, we recall that for a prime p.i. ring R, one considers (e.g., [R, p. 208]) the trace ring of R,

$$T(R) \equiv R[c_i(x) \mid 1 \leqq i \leqq n, x \in R]$$

of R, where $c_i(x)$ is the i-th coefficient of x in the Cayley–Hamilton equation $x^n - c_1(x)x^{n-1} \cdots \pm c_n(x) = 0$ and n^2 is the dimension of the quotient ring of R over its center. It is known (e.g. [Sch], [R]) that $T(R)$ is a finite module over its noetherian center, provided $R = \Lambda\{x_1, \ldots, x_k\}$, a prime P.I. ring and Λ is a central noetherian subring of R.

Let P be a height one prime ideal in R and $P_1, \ldots, P_r$ be all the prime ideals in $T(R)$ contracting to P (finite in number by Lemma 1). Let $p_i = P_i \cap Z(T(R))$, $i = 1, \ldots, r$, where $Z(T(R))$ is the center of $T(R)$. We say that P satisfies condition $(*)$ if the following implication holds,

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(*) If Q is a prime ideal in $T(R)$ and $Q \cap Z(T(R)) = p_j$ for some j, $1 \leqq j \leqq r$, then $Q = P_d$, for some $1 \leqq d \leqq r$.

We are now able to state our main result

THEOREM 1. *Let $R = \Lambda\{x_1, \ldots, x_k\}$ be a prime P.I. ring, Λ a central noetherian subring and P a prime ideal in R satisfying*

(1) height $(P) = 1$,

(2) *P/P^2 is a finitely generated right (left) R module.*

Then P is right (left) localizable iff P satisfies condition (*).

Theorem 1 is proved by showing that condition (*) implies (and in fact is equivalent) to another condition (**) as explained below. This is done in Theorem 2. We then use (**) in establishing the left localizability of P.

This generalizes one of the main results in [BW], replacing the noetherian assumption on R by (2). The price which is paid is firstly the severe restriction in (1), and secondly, the conclusion is not left–right symmetric, unlike the case in [BW]. These remarks actually raise several seemingly interesting questions;

QUESTION 1. Is (1) actually necessary for the conclusion of Theorem 1?

QUESTION 2. Assuming (1). Is (2) a necessary assumption in Theorem 1?. A related question is the following

QUESTION 3. Is there any left–right symmetric version of Theorem 1 (along the lines of [BW])?

The proof of the main results

In order to prove Theorem 1 we need to introduce another condition as follows. Let $S \equiv \mathscr{C}(P) \equiv \{r \in R \mid r \text{ is regular mode } P\}$. We say that P satisfies condition (**) if
for every $s \in S$ there exist $v_0, \ldots, v_{d-1} \in R$, $v_d \in S$ such that

$$(**) \qquad v_0 \det^d(s) + v_1 \det^{d-1}(s) + \cdots + v_{d-1} \det(s) + v_d = 0.$$

Important to our considerations is the following

THEOREM 2. *Let $R = \Lambda\{x_1, \ldots, x_k\}$ be a prime p.i. ring, Λ a central noetherian subring. Let P be a prime ideal in R such that* height$(P) = 1$ *and P satisfies* (*). *Then P satisfies* (**).

Before proving Theorem 2 we need several lemmas.

LEMMA 1. *Let $R = \Lambda\{x_1, \ldots, x_k\}$ be a prime p.i. ring, P a prime ideal in R with* height$(P) = 1$. *Let P' be a prime ideal in $T(R)$ such that $P' \cap R = P$. Then* height$(P') = 1$ *and there are only finitely many such P'.*

PROOF. $T(R)$ is noetherian (e.g. [R]), so let nil-radical $(PT(R)) = P_1 \cap \cdots \cap P_e$. If height$(P') > 1$ there are, by the principal ideal theorem as in [K, Thm. 144], infinitely many height one primes under it, each contracts non-trivially to R and so all must contract to P, violating the finiteness of k. Q.E.D.

We next introduce $R[P] \equiv R\{s^{-1} \mid s \in S, s^{-1} \in Q(R)\}$, the subring generated by R and $\{s^{-1}\}$, where $Q(R)$ is the Artinian quotient ring of R. Similarly we have $T(R)[P] = T(R)\{s^{-1} \mid s \in S\}$.

We next prove the following

LEMMA 2. *Let $R = \Lambda\{x_1, \ldots, x_k\}$ be a prime p.i. ring, P a prime ideal in R with* height$(P) = 1$. *Then*

(1) *$R[P]$, $T(R)[P]$ are prime rings with quotient ring $Q(R)$.*

(2) *$R[P]PR[P]$ is the unique non-zero prime ideal in $R[P]$ and $R[P]/R[P]PR[P] \cong Q(R/P)$, provided $R[P] \neq Q(R)$.*

PROOF. By $R \subset R[P] \subset T(R)[P] \subset Q(R)$, (1) is clearly established.

Let $V \neq \{0\}$ be a proper prime ideal in $R[P]$. By (1) $V \cap R \neq \{0\}$, moreover by [ArSc, Lemma 9.2] V ∩ R is a prime ideal in R. Clearly $V \cap R \subseteq P$. Now since height$(P) = 1$ we get $V \cap R = P$ and consequently $R[P]PR[P] \subset V$. We have

$$R/P \subseteq R[P]/R[P]PR[P] \equiv B.$$

Every non-zero divisor in R/P has a preimage in $S \equiv \mathscr{C}(P)$ and this is invertible in B. Let $\Delta = Z(R/P)$ and $\Delta^* \equiv \Delta \setminus \{0\}$. Since B is generated over R/P by inverses of elements in R/P we get that $\Delta \subset Z(B)$. Now as before each $\mathscr{J} \in \Delta$ has a preimage in S implying that $\mathscr{J}$ is invertible in B. Hence $(R/P)_{\Delta^*} \subseteq B$, that is $Q(R/P) \subseteq B$. But B is generated over R by inverses and they all live in $Q(R/P)$, so $Q(R/P) = B$ and consequently $R[P]PR[P]$ is a maximal ideal in $R[P]$ establishing (2).

PROPOSITION 3. *Let $R = \Lambda\{x_1, \ldots, x_k\}$ a prime p.i. ring and P a height one prime ideal in R satisfying $(*)$. Then*

$$T(R[P]) = T(R)_{\det(S)} = T(R)_\lambda,$$

where $S \equiv \mathscr{C}(P)$ and $\lambda \equiv Z(T) \setminus (p_1 \cup \cdots \cup p_r)$.

PROOF. We initially observe that the first equality is due to [BS, Lemma 1]. We next verify the inclusion $T(R)_\lambda \subseteq T(R)_{\det(S)}$. Indeed, let $z \in \lambda$ and suppose that z is not invertible in $T(R)_{\det(S)}$. Hence $z \in V$, a prime ideal in $T(R)_{\det(S)}$. We may assume (by the principal ideal theorem) that height$(V) = 1$. But $V = V_{0_{\det(S)}}$, where V_0 is a height 1 prime ideal in $T(R)$ implying that $z \in V \cap T(R) = V_0$. Now, $V_0 \cap R \subseteq P$, otherwise $V_0 \cap R \cap S \neq \emptyset$ implying the existence of $s \in V \cap R$, $s \in S$. But s is invertible in $T(R)_{\det(S)}$ (use the Cayley–Hamilton equation), a contradiction to the properness of V. Now height$(P) = 1$ implies $V_0 \cap R = P$ and consequently $V_0 = P_j$ for some j. Again we reach a contradiction to the choice of z since $z \in V_0 = P_j$. We remark that we did not use condition (*) for the previous inclusion. We now show that $T(R)_{\det(S)} \subseteq T(R)_\lambda$. By (*), $P_{1_\lambda}, \dots, P_{r_\lambda}$ are all the maximal ideals of $T(R)_\lambda$. So

$$R/P \subseteq T(R)_\lambda / \bigcap P_{i_\lambda} \cong T(R)_\lambda / P_{1_\lambda} \oplus \cdots \oplus T(R)_\lambda / P_{r_\lambda},$$

a semisimple Artinian ring. Let $s \in S$. If s is a zero divisor in $T(R)_\lambda / \bigcap P_{i_\lambda}$ it must be so in $T(R)_\lambda / P_{j_\lambda}$, for some j, $1 \leqq j \leqq r$. But $R/P \subseteq T(R)_\lambda / P_{j_\lambda}$ is a central extension and the latter is simple Artinian and of the same p.i. degree as R/P, a contradiction. So the image of S in $T(R)_\lambda / \bigcap P_{i_\lambda}$ consists of regular elements, hence of invertible elements. Now the Jacobson radical of $T(R)_\lambda$ is $\bigcap_{i=1}^{r} P_{i_\lambda}$, so S consists of invertible elements in $T(R)_\lambda$. Consequently $R[P] \subset T(R)_\lambda$. Thus $T(R[P]) \subset T(T(R)_\lambda) = T(R)_\lambda$ implying by [BS, Lemma 1] that $T(R)_{\det(S)} = T(R[P]) = T(R)_\lambda$. Q.E.D.

COROLLARY 4. *Let R, P be as in Proposition 3. Then*

$$\det(S) \cap \left(\bigcup_{i=1}^{r} P_i \right) = \emptyset, \qquad \text{where } S = \mathscr{C}(P).$$

PROOF. Let $s \in S$ so that $\det(s) \in P_j$ for some j. Hence $\det(s) \in p_j$. Now $\det(s)$ is invertible in $Z(T)_\lambda = Z(T(R)_\lambda)$, a contradiction to the properness of P_{j_λ}. Q.E.D.

COROLLARY 5. *Let R, P be as in the previous proposition. Then*

$$\text{Krull dim } R[P] = 1.$$

PROOF. We need (by Lemma 2) only show that $R[P] \neq Q(R)$. Suppose to the contrary that $R[P] = Q(R)$. Then $T(R[P]) = Q(R)$, but by Proposition 3, $T(R[P]) = T(R)_\lambda \neq Q(R)$, a contradiction.

LEMMA 6. *Let $R = \Lambda\{x_1, \dots, x_k\}$ be a prime p.i. ring and P a height one prime in R satisfying* $(*)$. *Then*

$$T(R)_{\det(S)} = T(R)[P].$$

PROOF. Clearly $T(R)_{\det(S)} \supseteq T(R)[P]$. In order to prove the opposite we need to show that $\det(S)$ consists of invertible elements in $T(R)[P]$. Suppose $\det(c)$ is not invertible in $T(R)[P]$ for some $c \in S$. So $\det(c) \in V$, a prime ideal in $T(R)[P]$. Now by Lemma 2 and Corollary 5, $V \cap R[P] = R[P]PR[P]$ and consequently $V \cap R = P$. Moreover, let $V_0 = V \cap T(R)$, then ([ArSc, Lemma 9.2]) V_0 is a prime ideal in $T(R)$ so that $V_0 \cap R = P$. Consequently $V_0 = P_j$ for some j, reaching, since $\det(c) \in P_j$, a contradiction to Corollary 4. Q.E.D.

LEMMA 7. *Let $R = \Lambda\{x_1, \dots, x_k\}$ a prime p.i. ring, P a height one prime ideal in R, and P' a prime ideal in $T(R)$ satisfying $P' \cap R = P$. Then, $T(R)_{\det(S)}/P'_{\det(S)}$ is a finite central extension of $Q(R/P)$, the quotient ring oj R/P.*

PROOF. By Lemma 1, Height$(P') = 1$. Now, $T(R)$ is finitely generated as a ring over R by a finite number of central elements, hence the same holds for $T(R)[P]$ over $R[P]$. Now by the previous Lemma $T(R)[P] = T(R)_{\det(S)}$ so $T(R)_{\det(S)}/P'_{\det(S)}$ is a finitely generated central extension of $R[P]/P'_{\det(S)} \cap R[P]$. But, we have $P'_{\det(S)} \cap R[P] = R[P]PR[P]$, so the result follows by $R[P]/R[P]PR[P] = Q(R/P)$, and by the Nullstellensatz applied to $T(R)_{\det(S)}/P'_{\det(S)}$, a finitely generated algebra over $Z((Q(R/P))$. Q.E.D.

The proof of Theorem 2

Let $P_1, \dots, P_r$, be the prime ideals in $T(R)$ satisfying $P_i \cap R = P$, $i = 1, \dots, r$. By Lemma 1, height$(P_i) = 1$, $i = 1, \dots, r$. By Lemma 7 $T(R)_{\det(S)}/P_{i_{\det(S)}}$ is a finite central extension of $Q(R/P)$, $i = 1, \dots, r$. Let $s \in S \equiv \mathscr{C}(P)$. By Corollary 4 $\det(s) \notin \bigcup_{i=1}^{r} P_i$. So $\det(s)$ has a non-zero divisor image in each of $T(R)_{\det(S)}/P_{i_{\det(S)}}$, $i = 1, \dots, r$; and therefore in

$$B \equiv T(R)_{\det(S)}/P_{1_{\det(S)}} \oplus \cdots \oplus T(R)_{\det(S)}/P_{r_{\det(S)}}.$$

We have, by the integrality of B over $Z(Q(R/P))$ (Lemma 7), the existence of $w, v_1, \dots, v_d \in R$, $w, v_d \in S$ so that

$$\overline{\det(s)}^{d} + \bar{v}_1\bar{w}^{-1}\,\overline{\det(s)}^{d-1} + \cdots + \bar{v}_d\bar{w}^{-1} = \bar{0},$$

where $\overline{\det(s)}$ is the image of $\det(s)$ in B. Equivalently,

$$w \det(s)^d + v_1 \det(s)^{d-1} + \cdots + v_d \in \bigcap_{i=1}^{r} P_i.$$

Let $I = \text{Conductor}(R, T(R)) \cap P$.

CLAIM. I is an ideal in $T(R)$.

PROOF. Let $D \equiv \text{Conductor}(R, T(R))$. Consequently $I = D \cap P$. Then $T(R)IT(R) \subseteq T(R)DT(R) \subseteq D$. Thus

$$T(R)IT(R) \subseteq D \cap T(R)PT(R) \subseteq R \cap T(R)PT(R) = P,$$

implying that $T(R)IT(R) \subseteq D \cap P = I$. Q.E.D.

Then since $\text{height}(P_i) = 1$, $i = 1, \ldots, r$, $I \subset P_i$ and P_i is minimal over I, $i = 1, \ldots, r$. Let $Q_1, \ldots, Q_e$ be the other minimal primes in $T(R)$ containing I. By (∗) there exists $t \in \bigcap_{i=1}^{e} Q_i \cap S$. Since

$$\text{nil-rad}\, I = P_1 \cap \cdots \cap P_r \cap Q_1 \cap \cdots \cap Q_e,$$

there exists a number f so that $(\text{rad}\, I)^f \subseteq I$ and so

$$[(w \det^d(s) + v_1 \det^{d-1}(s) + \cdots + v_d)t]^f \in I.$$

Consequently

$$a_0 \det^m(s) + a_1 \det^{m-1}(s) + \cdots + a_m \equiv x \in I,$$

where $a_0 = (wt)^f$, $a_1, \ldots, a_m = (v_d t)^f$ are all in R and $a_m \in S$. Finally $I \subset P$ implies that $a_m - x \in S$. Q.E.D.

LEMMA 8. *Let $R = \Lambda\{x_1, \ldots, x_k\}$ be a prime p.i. ring, P a height one prime ideal of R satisfying (∗∗). Let $I = \text{Conductor}(R, T(R)) \cap P$. Then for every integer m there exists an integer $f \equiv f(m)$ and $s \in S$ so that $P^f s \in I^m$, $sP^f \subseteq I^m$.*

PROOF. Firstly, as in the previous proof, I is an ideal in $T(R)$. Let $V_1, \ldots, V_r$ be the other minimal primes in R containing I.

We have $P^{f_1} V_{i_1}^{e_1} P^{f_2} V_{i_2}^{e_2} \cdots V_{i_p}^{e_p} P^{f_{p+1}} \subseteq I^m$. Since $V_i \neq P$ for $i = 1, \ldots, r$ we pick $t \in \bigcap_{i=1}^{r} V_i \cap S$. Let $e = \max\{e_i \mid i = 1, \ldots, p\}$ and $z \equiv t^e$. Thus

$$P^{f_1} z^\alpha P^{f_2} z^\beta P^{f_3} \cdots P^{f_p} z^\gamma P^{f_{p+1}} \subseteq I^m, \qquad \text{for all } \alpha, \beta, \ldots, \gamma \geqq 1.$$

Now since $\pm \det(z) = -c_{n-1}(z)z \cdots + c_1(z)z^{n-1} - z^n$ $(c_1(z) = \text{tr}(z))$, we have

$$\begin{aligned} &P^{f_1}z^{\alpha}P^{f_2}z^{\beta}P^{f_3}\cdots P^{f_p}\det(z)P^{f_{p+1}} \\ &\subseteq \sum_{i=1}^{n} P^{f_1}z^{\alpha}P^{f_2}z^{\beta}P^{f_3}\cdots P^{f_p}z^{i}P^{f_{p+1}}c_{n-i}(z) \\ &\subseteq \sum_{i=1}^{n} I^m c_{n-i}(z)\subseteq I^m, \end{aligned}$$

where the last inclusion is valid since I is an ideal in $T(R)$. Similarly

$$P^{f_1}z^{\alpha}P^{f_2}z^{\beta}P^{f_3}\cdots P^{f_p}\det^{\gamma}(z)P^{f_{p+1}}\subseteq I^m \qquad \text{for all } \alpha,\beta,\dots,\gamma\geqq 1.$$

Now by Theorem 2, taking $z=s$, there exist $v_0,\dots,v_d\in R$, $v_d\in S$ so that

$$v_0\det^d(s)+v_1\det^{d-1}(s)+\cdots+v_d=0.$$

Consequently

$$\begin{aligned} &P^{f_1}z^{\alpha}P^{f_2}z^{\beta}\cdots P^{f_p}P^{f_{p+1}}v_d \\ &\subseteq \sum_{i=0}^{d-1} P^{f_1}z^{\alpha}P^{f_2}z^{\beta}P^{f_3}\cdots P^{f_p}P^{f_{p+1}}\det(z)^{d-i}v_i \\ &=\sum_{i=0}^{d-1} P^{f_1}z^{\alpha}P^{f_2}z^{\beta}\cdots P^{f_p}\det^{d-i}(z)P^{f_{p+1}}v_i\subseteq I^m. \end{aligned}$$

We can continue the process and finish taking

$$f(m)=f_1+\cdots+f_{p+1} \quad\text{and}\quad s=(v_d)^{p+1}.$$

The argument for $sP^f\subseteq I^m$ is similar. Q.E.D.

Definitions. Let $J\subseteq P$ be an ideal in R. $J^{(i)}$, the i-th symbolic power of J, is defined by

$$J^{(i)}=\{x\in R \mid \text{there exists } s\in S, \text{ such that either } sx\in J^i \text{ or } xs\in J^i\}.$$

Remark. It is not clear that $J^{(i)}$, $i\geqq 1$, is an ideal in R (for an arbitrary ideal J).

Lemma 9. *Let I, P be as in Lemma 8. Then*

$$\begin{aligned} I^{(m)}&=\{x\in R \mid \textit{there exists } s\in S, sx\in I^m\} \\ &=\{x\in R \mid \textit{there exists } s\in S, xs\in I^m\} \\ &=\{x\in R \mid \textit{there exists } s\in S, \det(s)x\in I^m\}. \end{aligned}$$

PROOF. Let $x \in R$, $s \in S$ such that $sx \in I^m$. Hence

$$\det(s)x = \sum_{i=1} \pm c_{n-i}(s)s^i x \in \sum_{i=1}^{n} c_{n-i}(s)I^m \subseteq I^m,$$

since I is an ideal in $T(R)$. Conversely, say $\det(s)x \in I^m$ for some $s \in S$. Then, since I is an ideal in $T(R)$, $\det^\gamma(s)x \in I^m$ for all $\gamma \geqq 1$. By (**), there are $v_0, \ldots, v_{d-1} \in R$, $v_d \in S$, such that

$$v_0 \det^d(s) + v_1 \det^{d-1}(s) + \cdots + v_d = 0.$$

Consequently $v_d x = \sum_{i=1}^{d} - v_{d-i} \det^i(s)x \in I^m$. Consequently

$$\{x \in R \mid \text{there exists } s \in S, sx \in I^m\} = \{x \in R \mid \text{there exists } s \in S, \det(s)x \in I^m\}$$

and similarly is equal to $\{x \in R \mid \text{there exists } s \in S, xs \in I^m\}$. Clearly this implies that $I^{(m)}$ is equal to each one of the previous sets. Q.E.D.

COROLLARY 10. *$I^{(m)}$ is a two sided ideal in R.*

PROOF. Trivial by using Lemma 9.

LEMMA 11. *Let P, I, f, m be as in Lemma 8. Then we have*

$$P^f \subset P^{(f)} \subseteq I^{(m)} \subseteq P.$$

PROOF. Clearly $P^f \subset P^{(f)}$. Let $x \in P^{(f)}$. Hence, there exists $t \in S$ so that either $xt \in P^f$ of $tx \in P^f$. Say $xt \in P^f$. By Lemma 8, there exist $s \in S$ so that $P^f s \subseteq I^m$. Consequently, $x(ts) = (xt)s \in I^m$, that is $x \in I^{(m)}$. The possibility that $tx \in P^f$ is handled similarly using $sP^f \subseteq I^m$. So $P^{(f)} \subseteq I^{(m)}$ is established. Finally we shall show that $I^{(m)} \subseteq P$. Let $x \in I^{(m)}$ so $xs \in I^m \subset I \subset P$, for some $s \in S$. But $s \in S = \mathscr{C}(P)$ ensures that $x \in P$. Q.E.D.

LEMMA 12. *Let $\bar{R} = R/I^{(m)}$, $\bar{P} = P/I^{(m)}$. Then $\mathscr{C}(\bar{P}) \subseteq \mathscr{C}(\bar{0})$.*

PROOF. Let $\bar{a} \in \mathscr{C}(\bar{P})$, $\bar{a}\bar{x} = \bar{0}$. We shall show that $\bar{x} = \bar{0}$. We have $ax \in I^{(m)}$, $a \in \mathscr{C}(P)$. Let $t \in S$ satisfying $t(ax) \in I^m$, then $(ta)x \in I^m$ implies that $x \in I^{(m)}$, i.e. $\bar{x} = \bar{0}$. That $\bar{y}\bar{a} = \bar{0}$ implies $\bar{y} = \bar{0}$ is proved along the same lines. Q.E.D.

PROPOSITION 13. *Let $R = \Lambda\{x_1, \ldots, x_k\}$ be a prime p.i. ring, P a height one prime ideal in R such that P/P^2 is finitely generated as a right R module. Let $I =$ Conductor$(R, T(R)) \cap P$. Then, given $x \in R$, $s \in S$, there exist $y \in R$, $t \in S$ so that $xt - sy \in I^m$.*

PROOF. It is standard to show that P/P^i is right finitely generated for each i.

So, by Lemma 11, $\bar{R} = R/I^{(m)}$ has a unique minimal prime $\bar{P} = P/I^{(m)}$ and $\bar{P}$ is right finitely generated. Moreover, by Lemma 12, $\mathscr{C}(\bar{P}) \subseteq \mathscr{C}(\bar{0})$. So, by the non-noetherian version of Small's theorem appearing in [W], $\bar{R}$ has a right Artinian quotient ring. That is, there are $y_1 \in R$, $t_1 \in S$ so that $xt_1 - sy_1 \in I^{(m)}$. Let $t_2 \in S$ so that $(xt_1 - sy_1)t_2 \in I^m$ (Lemma 9). Take $t = t_1t_2$, $y = y_1t_2$. Q.E.D.

LEMMA 14. *Let R, P be as in Theorem 2. Let $S = \mathscr{C}(P)$ and $s \in S$. Then there exists a natural number m_1 so that if $a \in I^{m_1}$, there exists $t \in S$, $b \in I$ so that $at = sb$.*

PROOF. $I = IT(R) = IZ(T(R))$. So, since $T(R)$ is noetherian

$$I = g_1T(R) + \cdots + g_xT(R) = g_1RZ(T(R)) + \cdots + g_xRZ(T(R)),$$

where $g_i \in I \subset R$. Consequently, for any m, each element of I^m is a sum of terms of the form $g_{i1}r_1z_1g_{i2}r_2z_2 \cdots g_{im}r_mz_m$ where $r_i \in R$, $z_i \in Z(T(R))$, $i = 1, \ldots, m$. Now $s^{-1}I \subseteq I_{\det(S)}$, so $s^{-1}g_i = h_i(\det(y))^{-1}$, $h_i \in I$, for some $y \in S$. Equivalently $g_i \det(y) = sh_i$, $i = 1, \ldots, x$.

By (**) we have $v_0 \det^d(y) + \cdots + v_d = 0$, $v_i \in R$, $v_d \in S$. So

$$(g_{i1}r_1z_1 \cdots g_{id}r_dz_d)v_d = -(g_{i1}r_1z_1 \cdots g_{id}r_dz_d)(v_0 \det^d(y) + \cdots + v_{d-1}\det(y))$$

$$= -(sh_{i1}r_1z_1 \cdots sh_{id}r_dz_dv_0 + sh_{i1}r_1z_1 \cdots sh_{id-1}r_{d-1}z_{d-1}g_{id}r_dz_dv_1 + \cdots$$

$$+ sh_{i1}r_1z_1g_{i2}r_2z_2 \cdots g_{id}r_dz_dv_{d-1}) = sb,$$

where

$$b = -h_{i1}r_1z_1 \cdots sh_{id}r_dz_dv_0 - \cdots - h_{i1}r_1z_1g_{i2}r_2z_2 \cdots g_{id}r_dz_dv_{d-1}.$$

Now $b \in I$, since each summand contains $h_{i1} \in I$ and I is also an ideal in $T(R)$. By linearity the same holds for all elements of I^d. So pick $m_1 = d$, $t = v_d$, and b as above. Q.E.D

THE PROOF OF THEOREM 1. We establish first the sufficiency of (*). By Theorem 2, P satisfies (**). Let $x \in R$, $s \in S \equiv \mathscr{C}(P)$. We pick m_1 which satisfies the conclusion of Lemma 14. By Proposition 13 there exists $y \in R$, $t' \in S$ so that $xt' - sy \in I^{m_1}$. Let $a = xt' - ys \in I^{m_1}$ and b, t as in Lemma 14. Then $(xt' - sy)t = at = sb$. Equivalently, $xt't = s(yt + b)$ and the right Ore condition is clearly verified. To prove the necessity of condition (*) we use the same argument as in [BW]. Q.E.D.

References

[ArSc] M. Artin and W. Schelter, *Integral ring homomorphisms*, Advances in Math. **39** (3) (1981), 289–329.

[B] A. Braun, *An additivity principle for P.I. rings*, J. Algebra **96** (1985), 433–441.

[BS] A. Braun and L. Small, *Localization in prime noetherian P.I. rings*, Math. Z. **193** (1986), 323–330.

[BW] A. Braun and R. B. Warfield, Jr., *Symmetry and localization in Noetherian prime P.I. rings*, J. Algebra, to appear.

[G] A. W. Goldie, *Localization in non-commutative noetherian rings*, J. Algebra **5** (1967), 89–105.

[J] A. V. Jategaonkar, *Localization in Noetherian Rings*, London Mathematical Society Lecture Notes Series, 98, Cambridge, Cambridge University Press, 1986.

[K] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.

[R] L. H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, 1980.

[Sch] W. Schelter, *Integral extension of rings satisfying a polynomial identity*, J. Algebra **40** (1976), 245–257.

[W] R. B. Warfield, Jr., *Bezout rings and serial rings*, Commun. Algebra **7** (5) (1979), 533–545.